Mechanical Vibrations

Forced Vibration Systems

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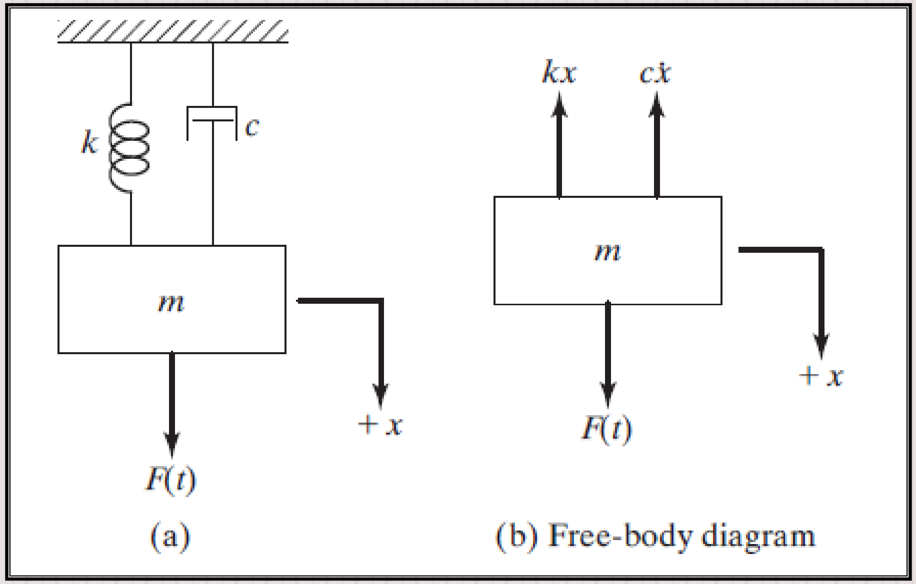
Harmonically Excited Vibration

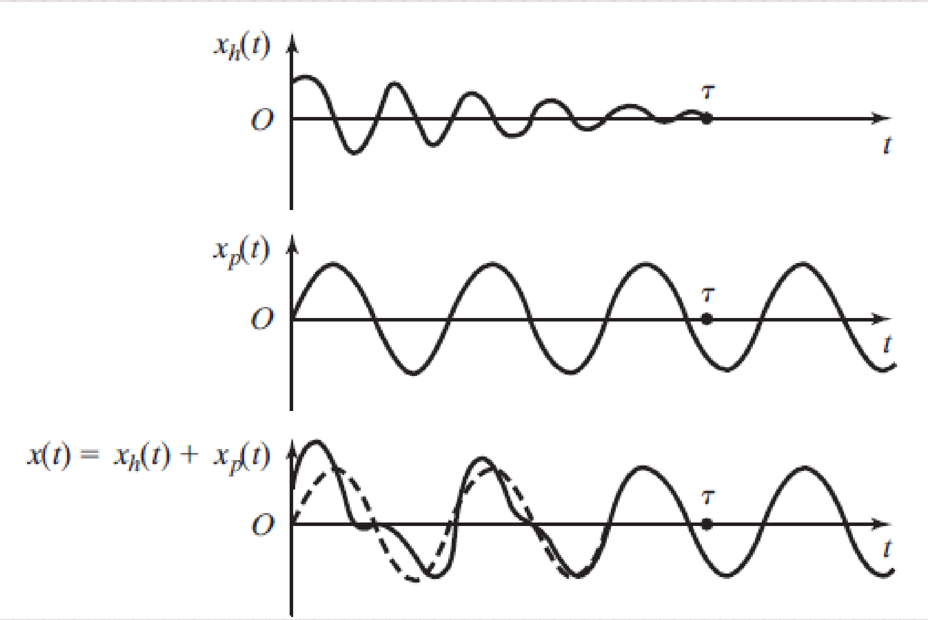
- Physical system



- Equation of motion: $m\ddot{x}+c\dot{x}+kx=F(t)$
- This equation is none – homogenous ODE
- The solution of this equation involves the solution of the homogenous part (discussed in the Ch.2) and a particular solution

Harmonically Excited Vibration

- Homogenous part of mathematical model represents a free vibration damped system: $m\ddot{x}+c\dot{x}+kx=0$
- As seen form the previous discussion for this type of vibration, the amplitude will eventually dies out.
- For steady state study of forced vibration, the particular solution is the only solution remains in the seen.



Un-damped Forced Vibration System

- **Un-damped system:** $m\ddot{x} + kx = F(t)$

- **Because the main cause of forced vibration in rotating machinery is the unbalance, the most convenient and simple form of forcing vibration $F(t) = F_o \cos(\omega t)$. So, the governing equation become:** $m\ddot{x} + kx = F_o \cos(\omega t) - - - Eq.1$

- **The homogenous solution (Free vibration):**

$$x(t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t)$$

- **to find the particular solution, assume: $x_p(t) = X\cos(\omega t)$…Eq.2 where: X is constant that donates the maximum amplitude of $x_p(t)$.**

Un-damped Forced Vibration System

- **Substitution of Eq.2 into Eq.1 to find X:** $X = \dfrac{F_o}{k - m\omega^2} = \dfrac{\delta_{st}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}$

where: $\delta_{st} = F_o/k$ denotes the deflection of mass (m) under the force F_o and it is called *static deflection*.

- **The general solution now become:**

$$x(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) + \frac{F_o}{k - m\omega^2}\cos(\omega t)$$

- **using the I.Cs:**

$$x(t=0) = x_o \Rightarrow C_1 = x_o - \frac{F_o}{k - m\omega^2}$$

$$\dot{x}(t=0) = \dot{x}_o \Rightarrow C_2 = \frac{\dot{x}_o}{\omega_n}$$

Un-damped Forced Vibration System

- **Substitution C_1 and C_2 in the general solution:**

$$x(t) = x_o - \frac{F_o}{k - m\omega^2}\cos(\omega_n t) + \frac{\dot{x}_o}{\omega_n}\sin(\omega_n t) + \frac{F_o}{k - m\omega^2}\cos(\omega t)$$

- **The ratio between maximum amplitude and the static deflection (X/δ_{st}) can be expressed as:**

Magnification factor

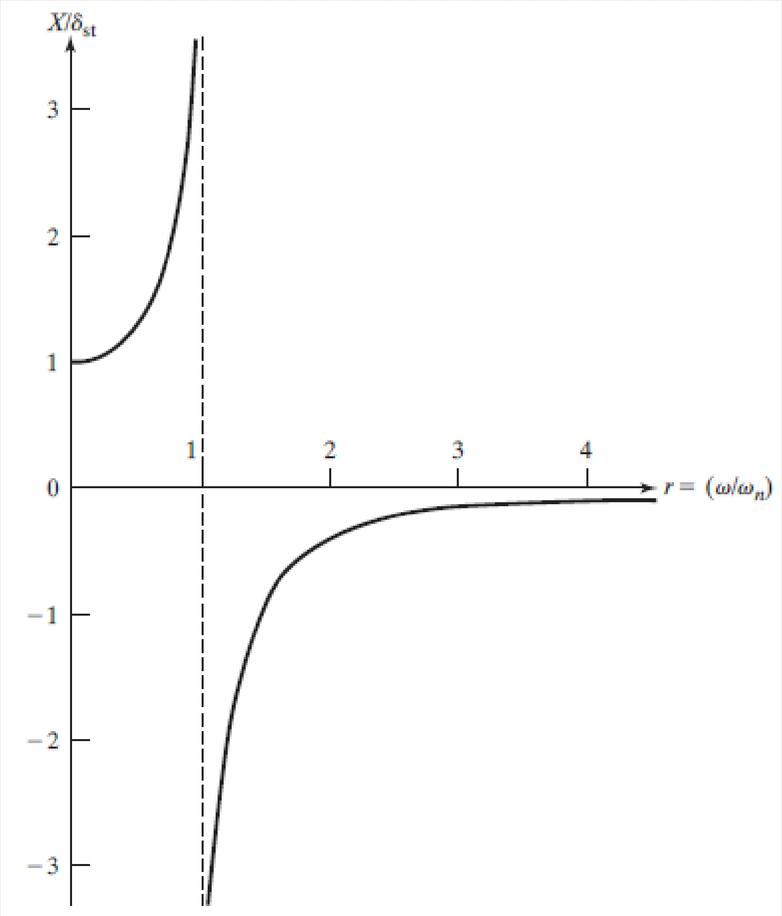
Frequency ratio

$$\frac{X}{\delta_{st}} = 1 \Big/ 1 - \left(\frac{\omega}{\omega_n}\right)^2 = 1/1 - r^2$$

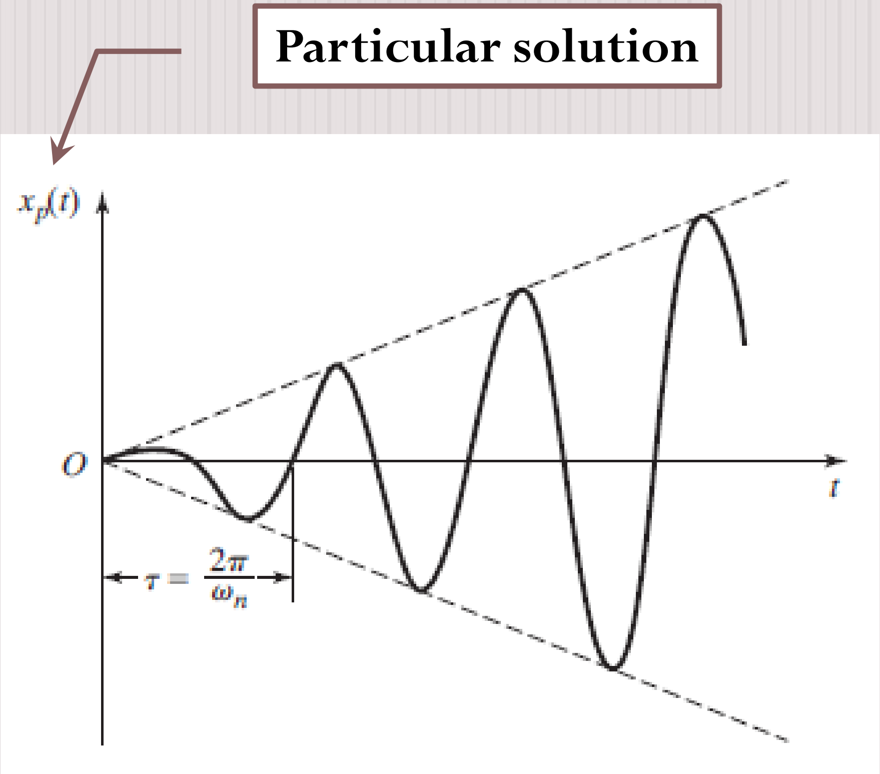
- **As you can see, there are three possible cases:**
  - **When 0<r<1**
  - **When r>1**
  - **When r = 1. this case is called resonance and the amplitude of vibration approaches the infinity.**

Resonance

All cases:



At resonance



Un-damped Forced Vibration System

Total response:

Homogenous solution

$$x(t) = A\cos(\omega_n t - \phi) + \frac{\delta_{st}}{1-r^2}\cos(\omega t) \quad for \;\; r < 1$$

$$x(t) = A\cos(\omega_n t - \phi) - \frac{\delta_{st}}{r^2-1}\cos(\omega t) \quad for \;\; r > 1$$

- **A and ϕ can be determined as in the previous analysis of free vibration:**

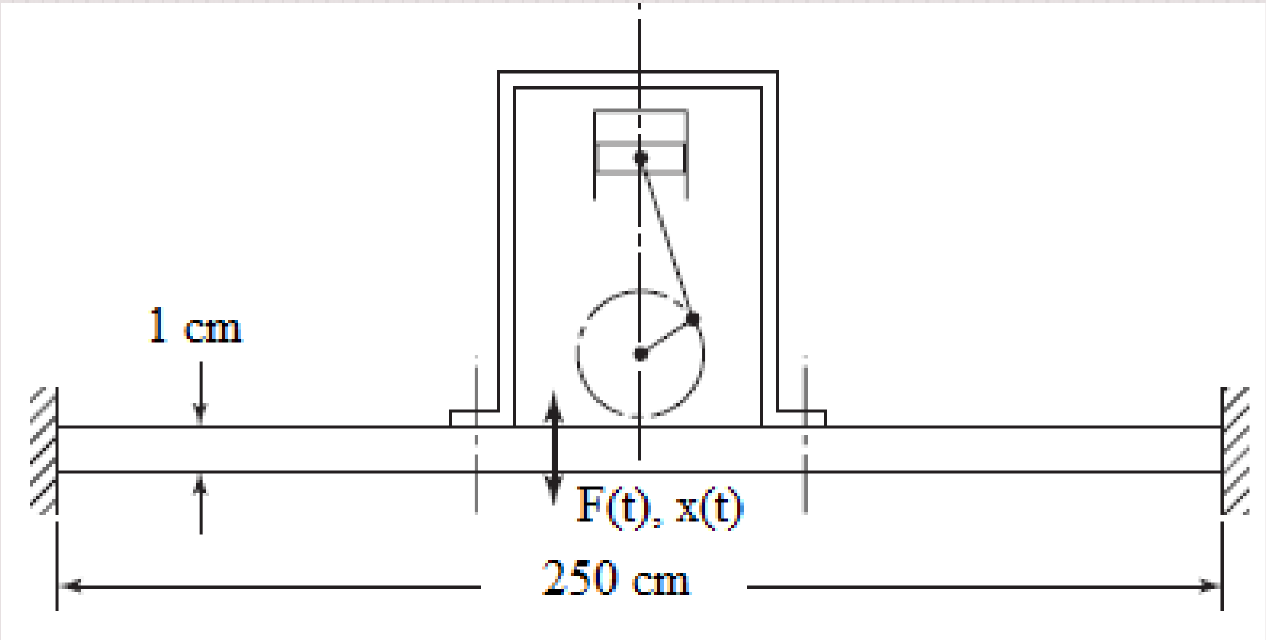
$$A = \sqrt{C_1^2 + C_2^2} = \sqrt{\left(x_o - \frac{F_o}{k - m\omega^2}\right)^2 + \left(\frac{\dot{x}_o}{\omega_n}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{C_2}{C_1}\right) = \tan^{-1}\left(\frac{\dot{x}_o / \omega_n}{x_o - F_o / k - m\omega^2}\right)$$

Un-damped Forced Vibration System

- **Example 3.1: Plate Supporting a Pump:**

A reciprocating pump, weighing 68 kg, is mounted at the middle of a steel plate of thickness 1 cm, width 50 cm, and length 250 cm. clamped along two edges as shown in Fig. During operation of the pump, the plate is subjected to a harmonic force, F(t) = 220 cos(62.832t) N. if E=200 Gpa, ***Find the amplitude of vibration of the plate.***



Un-damped Forced Vibration System

- **Example 3.1:** solution

- The plate can be modeled as fixed – fixed beam has the following stiffness:

$$k = \frac{192EI}{l^3}$$

$$But\ I = \frac{1}{12}bh^3 = \frac{1}{12}\left(50x10^{-2}\right)\left(1x10^{-2}\right)^3 = 41.667x10^{-9}\,m^4$$

$$So,\ k = \frac{192\left(200x10^{9}\right)\left(41.667x10^{-9}\right)}{\left(250x10^{-2}\right)^3} = 102{,}400.82\ N/m$$

- The maximum amplitude (X) is found as:

$$X = \frac{F_o}{k - m\omega^2} = \frac{220}{102{,}400.82 - 68(62.832)} = -1.32487\,mm$$

-ve means that the response is out of phase with excitation

Damped Forced Vibration System

- **Damped system:** $m\ddot{x} + c\dot{x} + kx = F_o \cos(\omega t)$
- **To find the particular solution, assume:** $x_p(t) = X\cos(\omega t - \phi)$.
- **Substitute the assumed solution into the governing equation and rearrange the terms:**

$$X\left[\left(k - m\omega^2\right)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)\right] = F_o \cos(\omega t)$$

- **Use the trigonometric relations**

$$\cos(\omega t - \phi) = \cos(\omega t)\cos(\phi) + \sin(\omega t)\sin(\phi)$$
$$\sin(\omega t - \phi) = \sin(\omega t)\cos(\phi) - \cos(\omega t)\sin(\phi)$$

- **Therefore:**

$$X\left[\left(k - m\omega^2\right)\cos(\phi) + c\omega \sin(\phi)\right] = F_o$$
$$X\left[\left(k - m\omega^2\right)\sin(\phi) - c\omega \cos(\phi)\right] = 0$$

- **Solve for X and ϕ:** $X = \dfrac{F_o}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}}$ $\quad \phi = \tan^{-1}\left(\dfrac{c\omega}{k - m\omega^2}\right)$

Damped Forced Vibration System

- Divide this equation $X = \dfrac{F_o}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$ by k and make the following substations:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \xi = \frac{c}{c_c} \qquad \delta_{st} = \frac{F_o}{k} \qquad r = \frac{\omega}{\omega_n}$$

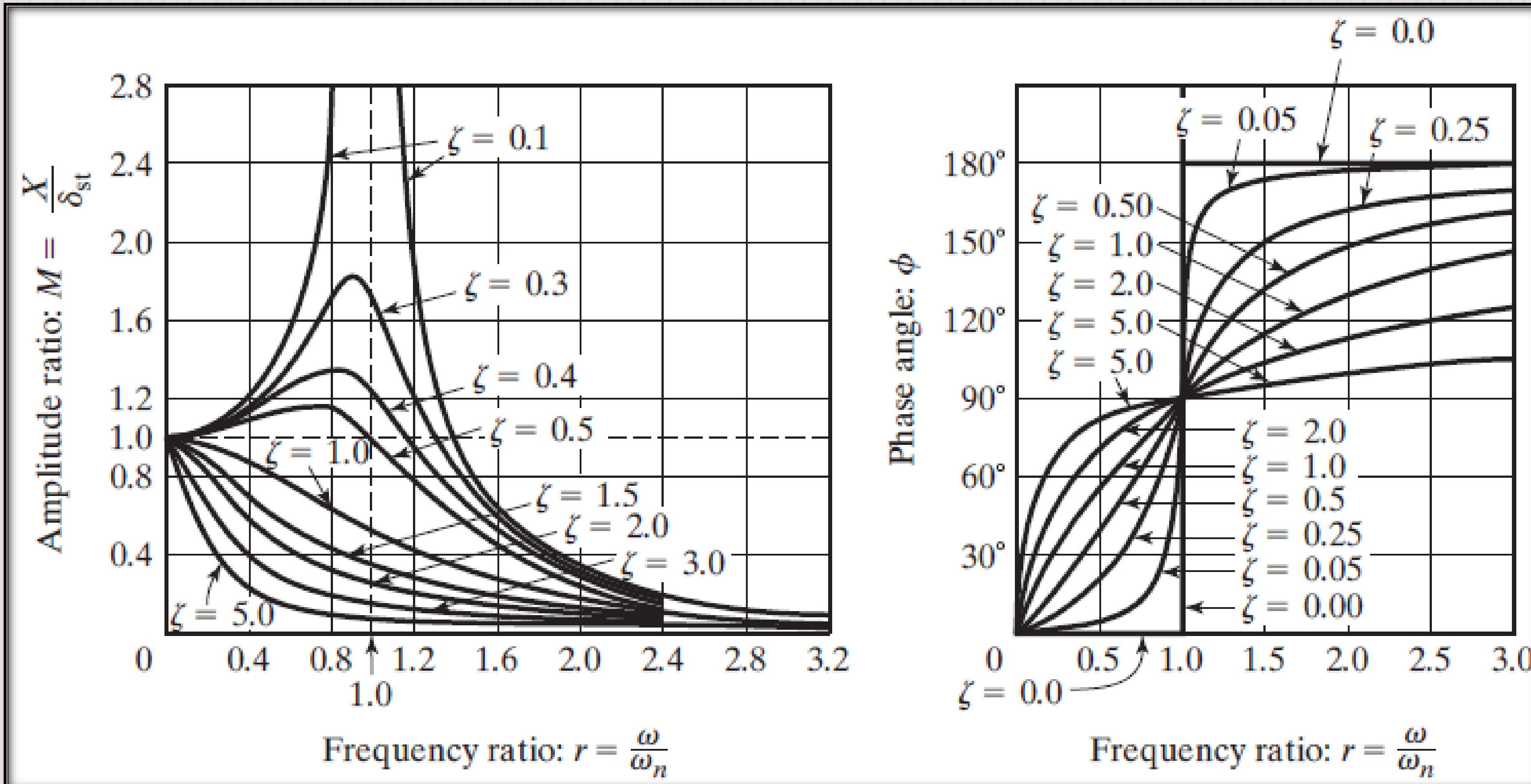
- You will eventually get:

$$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{\left(1 - r^2\right)^2 + \left(2\xi\, r\right)^2}} \quad \text{and } \phi = \tan^{-1}\left(\frac{2\xi\, r}{1 - r^2}\right)$$

Also called amplitude ratio (M)

Damped Forced Vibration System

- **Graphical representation for X and ϕ.**



Damped Forced Vibration System

- Notes on the graphical representation for X.

  - For $\zeta = 0$, the system is reduced to become un-damped.
  - for any amount of (ζ) ; $\zeta > 0$, the amplitude of vibration decreases (i.e. reduction in the magnification factor *M*). This is correct for any value of *r*.
  - For the case of $r = 0$, the magnification factor equal 1.
  - The amplitude of the forced vibration approaches zero when the frequency ration approaches the infinity (i.e. $M \to 0$ when $r \to \infty$)

Damped Forced Vibration System

- Notes on the graphical representation for ϕ.

  - For $\zeta = 0$, the phase angle is zero for $0<r<1$ and 180^o for $r>1$.
  - For any amount of (ζ) ; $\zeta > 0$ and $0<r<1$, $0^o<\phi<90^o$.
  - For $\zeta > 0$ and $r>1$, $90^o<\phi<180^o$.
  - For (ζ) ; $\zeta > 0$ and $r=1$, $\phi= 90^o$.
  - For (ζ) ; $\zeta > 0$ and $r>>1$, ϕ approaches 180^o.

Damped Forced Vibration System

Total response

- **The total response is** $x(t) = x_h(t) + x_p(t)$
- **For under-damped system, the general solution is given as:**

$$x(t) = Xe^{-\xi\omega_n t}\{\cos(\omega_d t - \phi)\} + X\cos(\omega t - \phi) \quad \textbf{where: } \omega_d = \sqrt{1-\xi^2}\,\omega_n$$

- **Assume the I.Cs:** $x(t=0) = x_o$ and $\dot{x}(t=0) = \dot{x}_o$ **and substitute it in the general solution:**

$$\left.\begin{aligned} x_o &= X_o\cos(\phi_o) + X\cos(\phi) \\ \dot{x}_o &= -\xi\omega_n X_o\cos(\phi_o) + \omega_d X_o\sin(\phi_o) + \omega X\sin(\phi) \end{aligned}\right\} --- Eq.1$$

Damped Forced Vibration System

Total response

- **Solve the Eq.1 to find X_o and ϕ_o:**

$$X_o = \left[\left(x_o - X\cos(\phi)\right)^2 + \frac{1}{\omega_d^2}\left(\xi\omega_n x_o + \dot{x}_o - \xi\omega_n X_o\cos(\phi_o) - \omega X\sin(\phi)\right)^2 \right]$$

$$\tan(\phi_o) = \frac{\left(\xi\omega_n x_o + \dot{x}_o - \xi\omega_n X_o\cos(\phi_o) - \omega X\sin(\phi)\right)}{\omega_d\left(x_o - X\cos(\phi)\right)}$$

Damped Forced Vibration System

Example 3.2:

Find the total response of a single-degree-of-freedom system with m = 10 kg, c = 20 N-s/m, k=4000 N/m, x_o = 0.01m and $\dot{x}_o = 0$ under the following conditions:

a. An external force F(t) = $F_o\cos(\omega t)$ acts on the system with F_o = 100 N and ω = 10 rad/sec

b. Free vibration condition : F(t) = 0

Solution

a. From the given data

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 rad/s$$

$$\xi = \frac{c}{2m\omega_n} = \frac{20}{2(10)(20)} = 0.05$$

$$\omega_d = \sqrt{1-\xi^2}\,\omega_n$$

$$\omega_d = \sqrt{1-0.05^2}\,(20)$$

$$\omega_d = 19.975 rad/s$$

$$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$$

Damped Forced Vibration System

Example 3.2:

Solution

$$\delta_{st} = \frac{F_o}{k} = \frac{100}{4000} = 0.025m \longrightarrow X = \frac{\delta_{st}}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}} = 0.3326m$$

$$\phi = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right) = 3.814^o$$

- **Using I.Cs to find X_o and ϕ_o**

$$X_o = \left[\left(x_o - X\cos(\phi)\right)^2 + \frac{1}{\omega_d^2}\left(\xi\omega_n x_o + \dot{x}_o - \xi\omega_n X_o\cos(\phi_o) - \omega X\sin(\phi)\right)^2\right] = 0.0233$$

$$\tan(\phi_o) = \frac{\left(\xi\omega_n x_o + \dot{x}_o - \xi\omega_n X_o\cos(\phi_o) - \omega X\sin(\phi)\right)}{\omega_d\left(x_o - X\cos(\phi)\right)} \Rightarrow \phi = 5.587^o$$

Damped Forced Vibration System

Example 3.2:

Solution

b. For free vibration: $x(t) = Xe^{-\xi\omega_n t}\left\{\cos\left(\sqrt{1-\xi^2}\,\omega_n t - \phi\right)\right\}$

$$X = \frac{\sqrt{x_o^2\omega_n^2 + \dot{x}_o^2 + 2x_o\,\dot{x}_o\,\xi\omega_n}}{\sqrt{1-\xi^2}\,\omega_n} = 0.010012$$

$$\phi = \tan^{-1}\left[\frac{\dot{x}_o + \xi\omega_n x_o}{x_o\omega_n\sqrt{1-\xi^2}}\right] = 2.866^o$$

Substitute the values of X and ϕ into the general solution :

$$x(t) = 0.010012e^{-t}\left\{\cos\left(19.975t - \phi\right)\right\}$$

Base excitation

Example 3.2:

Solution

b. For free vibration: $x(t) = Xe^{-\xi\omega_n t}\left\{\cos\left(\sqrt{1-\xi^2}\,\omega_n t - \phi\right)\right\}$

$$X = \frac{\sqrt{x_o^2\omega_n^2 + \dot{x}_o^2 + 2x_o\dot{x}_o\xi\omega_n}}{\sqrt{1-\xi^2}\,\omega_n} = 0.010012$$

$$\phi = \tan^{-1}\left[\frac{\dot{x}_o + \xi\omega_n x_o}{x_o\omega_n\sqrt{1-\xi^2}}\right] = 2.866^o$$

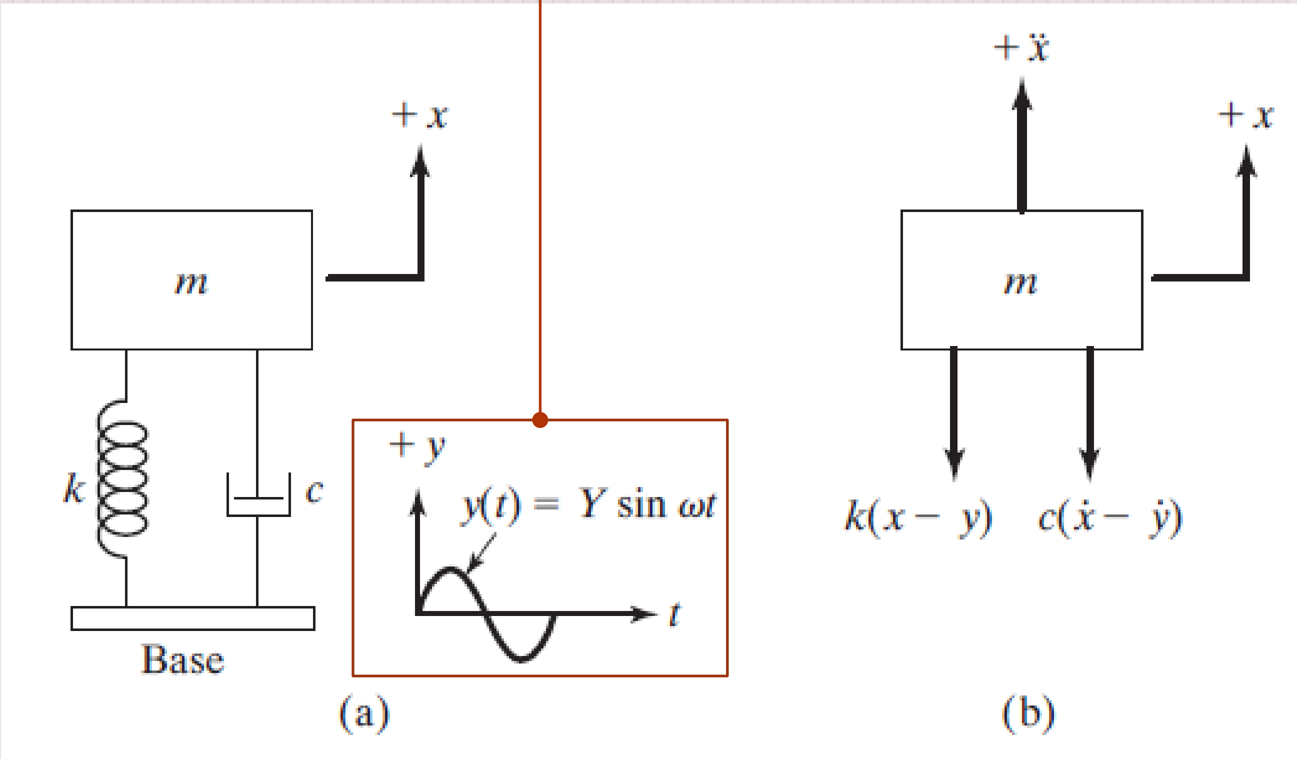
Substitute the values of X and ϕ into the general solution :

$$x(t) = 0.010012e^{-t}\left\{\cos(19.975t - \phi)\right\}$$

Forced Vibration System under Base Excitation

Physical system:

The forcing function for the base excitation



Mathematical model: $m\ddot{x}+c\left(\dot{x}-\dot{y}\right)+k\left(x-y\right)=0$

Forced Vibration System under Base Excitation

Substitute the forcing function into the math. Model:

$$m\ddot{x} + c\dot{x} + kx = kY\sin(\omega t) + c\omega Y\cos(\omega t) = A\sin(\omega t - \alpha)$$

Where:

$$A = Y\sqrt{k^2 - (c\omega)^2}$$

$$\alpha = \tan^{-1}\left(-\frac{c\omega}{k}\right)$$

This is harmonic excitation force

The particular solution: $x_p(t) = X\cos(\omega t - \phi)$.

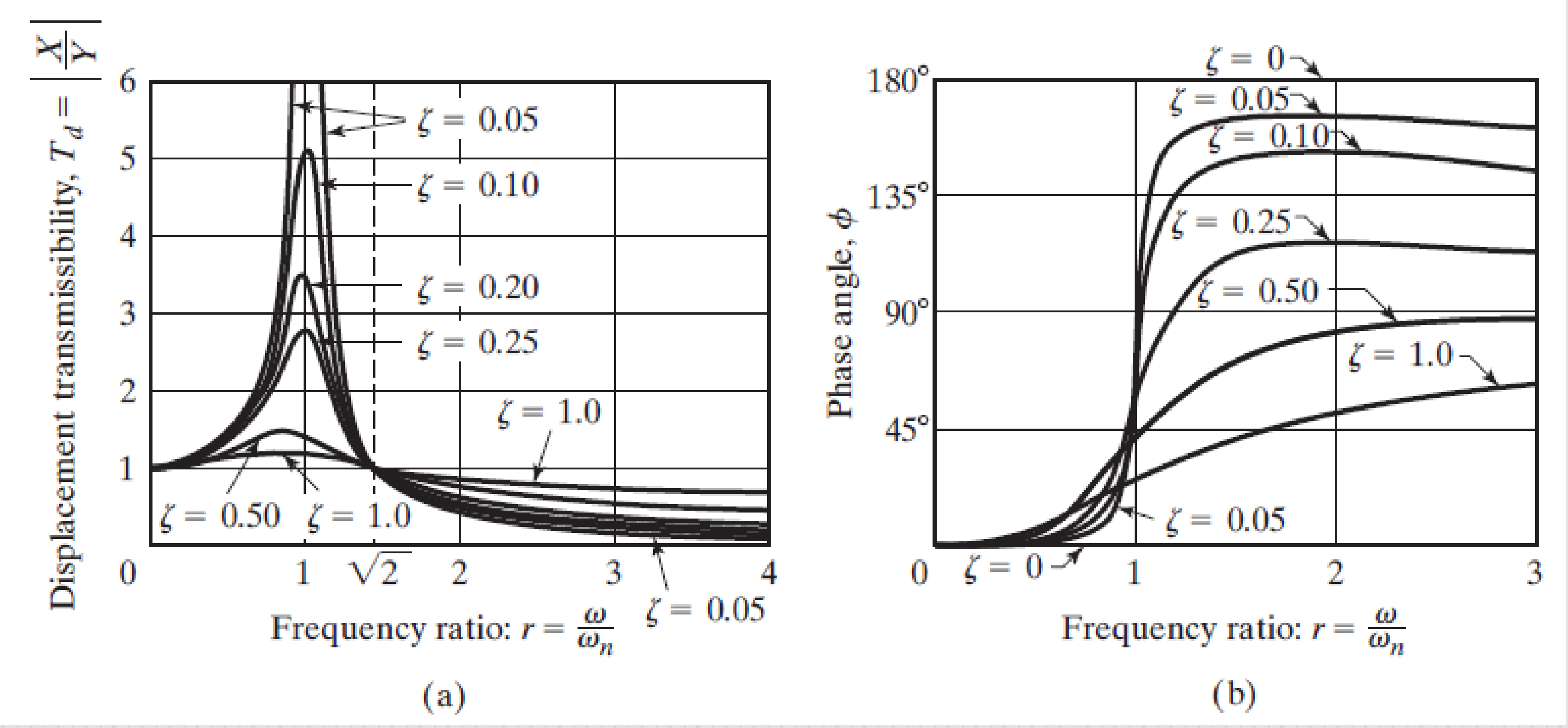
Displacement transmissibility

$$\frac{X}{Y} = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}} = \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2}}$$

$$\phi = \tan^{-1}\left(\frac{mc\omega^3}{k(k - m\omega^2) + (c\omega)^2}\right) = \tan^{-1}\left(\frac{2\xi r^3}{1 + (4\xi^2 - 1)r^2}\right)$$

Forced Vibration System under Base Excitation

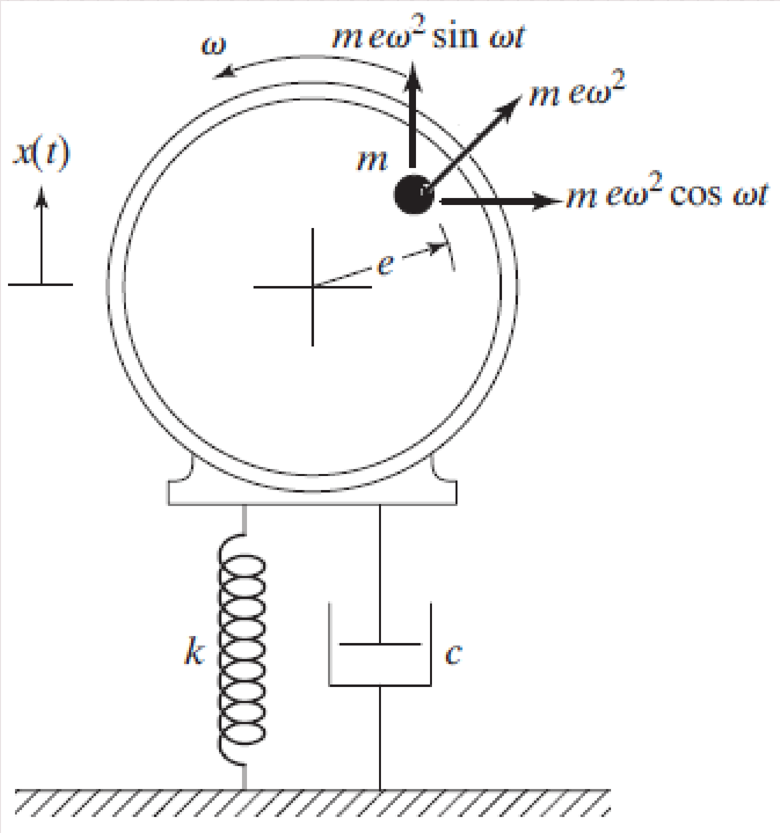
Graphical representation for X and ϕ.



Forced Vibration System due to Unbalance

Assume the following general case: a harmonic excited vibrating system which discussed before.

- The centrifugal force equal $me\omega^2$
- The vertical component $(me\omega^2 \sin(\omega t))$ is the effective one because the direction of motion is vertical.
- the vertical component can be moved to the center of rotation due to vector definition.
- The equation of motion can derived as: $m\ddot{x} + c\dot{x} + kx = me\omega^2 \sin(\omega t)$
- The derived equation of monition is



Forced Vibration System General Periodic force

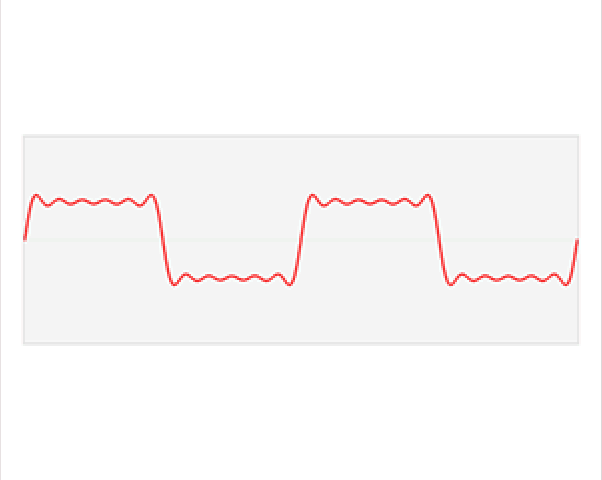
- **In some cases, the external force F(t) is periodic with period: τ=2π/ω.**
- **Periodic force can expanded in Fourier series as:**

$$F(t)=\frac{a_o}{2}+\sum_{j=1}^{\infty}a_j\cos(j\omega t)+\sum_{j=1}^{\infty}b_j\sin(j\omega t)$$

Where:

$$a_j=\frac{2}{\tau}\int_0^{\tau}F(t)\cos(j\omega t).dt$$

$$b_j=\frac{2}{\tau}\int_0^{\tau}F(t)\sin(j\omega t).dt$$



- **The order of the expansion is (j)**
- **As the number of terms increases, as the accuracy of expanding increases.**